7

SYSTEM OF PARTICLES AND ROTATIONAL MOTION

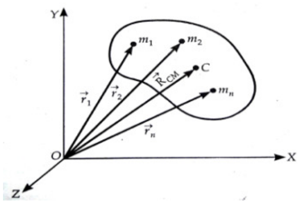
- **Rigid Body**

A body is said to be rigid body if the distance between its particles is always the same.

- **Centre of mass**

Centre of mass of a system of particles is the point where the entire mass of the system is assumed to be concentrated. When the system moves under an external force, the centre of mass moves as if the force is acting it.

- **Position vector of centre of mass of a system of n particles**



Consider a system consisting of particles of mass m_1 *and* m_2 …,m_n with position vectors $\vec{r_1}, \vec{r_2}$ …..$\vec{r_n}$with respect to an origin O. The position vector of centre of mass is $\overrightarrow{R_{cm}}$.

Then

$m_1\vec{r_1} + m_2\,\vec{r_2} + \ldots\, m_n\vec{r_n} \;=\; (m_1 \;+ m_2 + \ldots\, m_n)\,\overrightarrow{R_{cm}}$

$\overrightarrow{R_{cm}} \;= \frac{m_1\vec{r_1} + m_2\,\vec{r_2} + \ldots.. + m_n\vec{r_n}}{m_1 \;+ m_2 + \ldots.. + m_n} \;=\; \frac{m_1\vec{r_1} + m_2\,\vec{r_2} + \ldots\, m_n\vec{r_n}}{M}$

(where M = $m_1 \;+ m_2 + \ldots\, m_n$)

This is the expression for the position vector of centre of mass.

- **Velocity of centre of mass**

$\overrightarrow{v_{cm}} \;= \frac{d\overrightarrow{R_{cm}}}{dt}$

$\overrightarrow{v_{cm}} \;= \frac{m_1\vec{v_1} + m_2\,\vec{v_2} + \ldots + m_n\vec{v_n}}{M}$

- **Acceleration of centre of mass**

$$\overrightarrow{a_{cm}} = \frac{d}{dt} v_{cm}$$
$$\overrightarrow{a_{cm}} = \frac{m_1\overrightarrow{a_1} + m_2\overrightarrow{a_2} + \ldots + m_n\overrightarrow{a_n}}{M}$$

- **Some applications of the concept of centre of mass**

  - Earth – moon system:

    The earth moves round the sun. The moon moves round the earth. The centre of mass of earth- moon system moves round the sun in elliptical orbit.

  - Explosion of a shell:

    Consider a shell which is projected to air. If it is not exploding, its path will be a parabola. If it is exploding, the centre of mass of the fragments still move in parabolic path. (Internal forces causing explosion can not affect the motion of centre of mass).

  - Decay of a radioactive nucleus.

    Consider a radioactive nucleus at rest. If it disintegrates in two fragments, the product fragments move away from each other . The centre of mass of the system still remains at rest. (as the disintegration is due to internal force)

- **Centre of mass of some rigid bodies**

  - Rod - midpoint
  - Ring - Centre
  - Circular disc –Centre
  - Sphere- Centre
  - Rectangular lamina - Point of intersection of diagonals
  - Triangular lamina - Point of intersection of medians
  - Cylinder - Midpoint of axis
  - Right circular cone – On the axis at a distance of $\frac{h}{4}$ from base

- The tendency of a rotating body to continue in the same state is called **rotational inertia.**

- Moment of inertia is a measure of rotational inertia.

  Moment of inertia (I) of a particle about an axis of rotation is the product of mass of the particle and the square of distance between the axis of rotation and the particle.

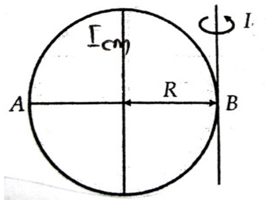
  I = mR^2 Unit: kgm^2

- **Moment of inertia of rigid body**

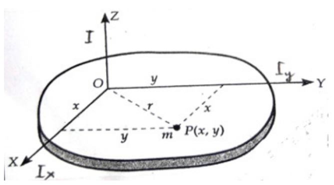
Consider a rigid body made of particles with masses m_1 ,m_2 ,….. at distance r_1 ,r_2 ,… from the axis of rotation.
Moment of inertia of the body I = $m_1 r_1^2 + m_2 r_2^2$ +…..+ $m_n r_n^2$
I = $\sum_{i=1}^{n} m_i r_i^2$

- **Parallel axis theorem**

The moment of inertia of a body about any axis is the sum of the moment of inertia of the body about a parallel axis through centre of mass and the product of the mass of the body and the square of distance between the axes.
Here I = I_{cm} + Ma^2 where M is the of the body, I_{cm} is the moment of inertia about the centre of mass and 'a' is the distance between the axes.

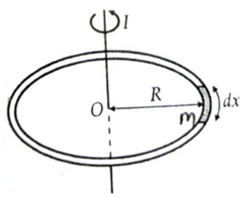


- **Perpendicular axis theorem**



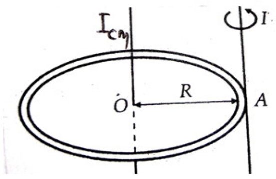
The moment of inertia of a plane lamina about an axis perpendicular to its plane is the sum of moment of inertia about two mutually perpendicular axes on the lamina which meet at a point where the perpendicular axis meets the lamina.
Here I = $I_X + I_Y$

- **Moment of inertia of a ring about an axis through its centre and perpendicular to its plane**



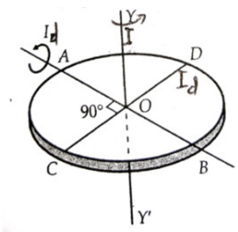
Consider a ring of mass M and radius R. It is rotated about an axis through its centre and perpendicular to its plane.
Consider a small element of mass 'm' at distance R from the axis.
Moment of inertia of the element = $m\,R^2$
Moment of inertia of the ring = sum of moment of inertia of all such element on the ring
Moment of inertia of the ring = $\sum m\,R^2$ = MR^2

- **Moment of inertia of a ring about a tangent perpendicular to its plane**



To calculate the moment of inertia, we use a parallel axis theorem .We consider a parallel axis through the centre of mass.
Moment of inertia about the centre of mass I_{cm}= $M\,R^2$
I = I_{cm} + Ma^2
a = R
I = $M\,R^2$ + $M\,R^2$ = 2$M\,R^2$

- **Moment of inertia of ring about a diameter**



To calculate the moment of inertia of a ring about a diameter, we consider two mutually perpendicular diameters. Moment of inertia about both these diameters will be equal (i_d)

Applying perpendicular axis theorem,

$I\ = I_d$+I_d

$M\,R^2$ = I_d+I_d

$M\,R^2$ = 2 I_d

I_d = $\frac{M\,R^2}{2}$

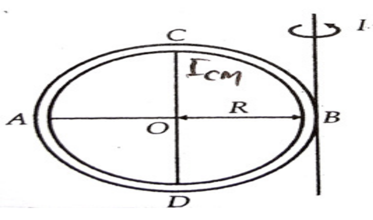
- **Moment of inertia of a ring about a tangent in the plane of the ring**

Moment of inertia about the tangent,

I = I_{cm} + Ma^2

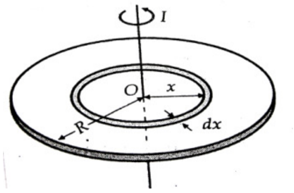
I = $\frac{M\,R^2}{2}$ + $M\,R^2$

I = $\frac{3M\,R^2}{2}$



- **Moment of inertia of disc**

❖ About an axis through centre and perpendicular to its plane.



Consider a circular disc of mass M and radius R.

Mass/unit area = $\frac{M}{\pi R^2}$

To calculate moment of inertia, we consider a small elementary ring of thickness dx at a distance x from the centre.

Area of the elementary ring = 2 $\pi x dx$

Mass of the elementary ring = 2 $\pi x dx \times \frac{M}{\pi R^2}$

Moment of inertia of elementary ring about an axis through centre and perpendicular to its plane = $M\ R^2$

=2 $\pi x dx \times \frac{M}{\pi R^2} \times x^2$ = $\frac{M}{\pi R^2}\, 2\pi x^3 dx$

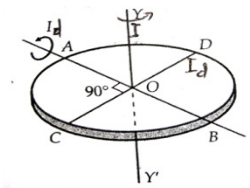
Moment of inertia of the disc = $\int_0^R \frac{M}{\pi R^2} 2\pi x^3 dx$

= $\frac{M}{\pi R^2}\, 2\, \pi \int_0^R x^3\, dx$

= $\frac{2\ M}{R^2}\left[\frac{x^4}{4}\right]_0^R$ = $\frac{2\ M}{4\ R^2}[R^4 - O^4]$

OR I = $\frac{M\ R^2}{2}$

❖ Moment of inertia of disc about a diameter



To calculate the moment of inertia of disc about a diameter , we consider two mutually perpendicular diameters. Moment of inertia about both these diameter will be equal (I_d)

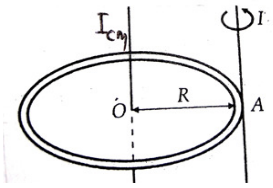
Applying perpendicular axis theorem,

I = I_d+I_d

$\frac{M\ R^2}{2}$ = 2I_d

OR I_d = $\frac{M\ R^2}{4}$

❖ Moment of inertia of a disc about a tangent perpendicular to its plane.



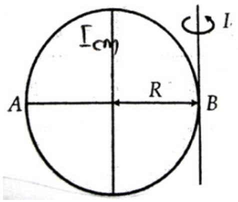
To calculate the moment of inertia, we use a parallel axis theorem. We consider a parallel axis through the centre of mass.

Moment of inertia about centre of mass $I_{cm} = \frac{M R^2}{2}$

Moment of inertia about the tangent I = $I_{cm} + Ma^2$ Here a = R

$I = \frac{M R^2}{2} + M R^2 = \frac{3 M R^2}{2}$

❖ Moment of inertia of a disc a tangent in the plane of the ring.



To calculate the moment of inertia, we use a parallel axis theorem. We consider a parallel axis through the centre of mass.

Moment of inertia about the tangent,

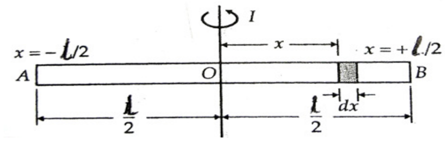
I = $I_{cm} + Ma^2$

$I = \frac{M R^2}{4} + MR^2$

$I = \frac{5M R^2}{4}$

- **Moment of inertia of a uniform rod**

❖ About an axis through its centre and perpendicular to its length.



Consider a uniform rod of length l and mass M.

Mass/ unit length = $\frac{M}{l}$

Consider an element of length dx at a distance x from the axis. When the rod is rotated, the element describes a ring of radius x.

Mass of the element (ring)=$\frac{M}{l} dx$

Moment of inertia of the element (ring) $\frac{M}{l} dx\, x^2$

Moment of inertia of the rod = $\int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{M}{l} x^2 dx$ = $\frac{M}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 dx$ = $\frac{M}{l}\left[\frac{x^3}{3}\right]_{-\frac{l}{2}}^{\frac{l}{2}}$

= $\frac{M}{3l}[x^3]_{-\frac{l}{2}}^{\frac{l}{2}}$ = $\frac{M}{3\,l}\left[\left(\frac{l}{2}\right)^3 - \left(-\frac{l}{2}\right)^3\right]$

= $\frac{M}{3\,l}\left[\frac{l^3}{8}+\frac{l^3}{8}\right]$ = $\frac{M}{3\,l}\left[\frac{2\,l^3}{8}\right]$ = $\frac{M\,l^2}{12}$

- ❖ About an axis through its end and perpendicular to its length

  To calculate the moment of inertia, we use a parallel axis theorem. We consider a parallel axis through the centre of mass.

  Moment of inertia about centre of mass = I_{cm} = $\frac{M\,l^2}{12}$

  Moment of inertia about an axis through its end and perpendicular to its length = $I_{cm} + Ma^2$ (Here a = $\frac{l}{2}$)

  Moment of inertia about an axis through its end and perpendicular to its length = $\frac{M\,l^2}{12}$ + M $\left(\frac{l}{2}\right)^2$

  = $\frac{M\,l^2}{12}$ + M$\frac{l^2}{4}$ = $\frac{M\,l^2}{12}$ + M$\frac{3l^2}{12}$ = $\frac{4\,M\,l^2}{12}$ = $\frac{M\,l^2}{3}$

- **Moment of inertia of some regular shaped bodies**

  Hollow cylinder of radius R about its axis: MR^2

  Solid cylinder of radius R about it axis: $\frac{M\,R^2}{2}$

  Hollow sphere (spherical shell) of radius R about its diameter: $\frac{2\,M\,R^2}{3}$

  Solid sphere of radius R about its diameter: $\frac{2\,M\,R^2}{5}$

- The sliding motion of a block along an inclined plane is example for pure translation motion. Rolling motion of a cylinder along an inclined plane is a combination of translational and rotational motion.

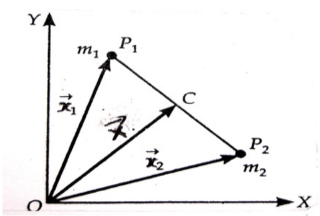
- **Example for rotational motion where the axis is not fixed**

  - ❖ The spinning motion of a top . Here the axis of the top sweeps a conical shape around the vertical about a point on the ground.
  - ❖ The oscillatory motion of a table fan . Here the axis of rotation oscillates in a horizontal plane about the vertical .

- The motion of rigid body which is not pivoted or fixed is pure translation or a combination of translation and rotation.

  The motion of a body which is pivoted or fixed is rotation.

- **The centre of mass of a two particle system where the particles are along a line**



Consider a two particles system where the particles are along a line. Let the distance from origin to the particles be $x_1$and x_2 . The distance to the centre of mass is X . Let the masses of the particles are identical, $m_1 = m_2$. Then,

$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$. If the particles are identical, $m_1 = m_2 = m$

$\therefore X = \frac{m x_1 + m x_2}{m + m} = \frac{m(x_1 + x_2)}{2m} = \frac{(x_1 + x_2)}{2}$

ie,If the particles are identical, the centre of mass of the system will be at the midpoint of the line joining the two masses.

For the system of three particle , $X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$.

If the particles are identical, $m_1 = m_2 = m_3 = m$,

Then $X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{3m} = \frac{m(x_1 + x_2 + x_3)}{3m} = \frac{(x_1 + x_2 + x_3)}{3}$ ie, The centre of mass is the centroid of the triangle formed by $m_1, m_2 \text{and} m_3$ at its vertices.

- **For an n particle system, (where particles may not be along a line)**

The X coordinate of the centre of mass $X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots}{m_1 + m_2 + m_3 +}$

The Y coordinate of the centre of mass $Y = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$

Generally , $R = \frac{m_1 r_1 + m_2 r_2 + m_3 r_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum m_i r_i}{M}$

$$\sum m_i \rightarrow \int dm \sum m_i r_i \rightarrow \int r dm$$

$\therefore R = \frac{\sum m_i r_i}{M} = \frac{\int r dm}{M}$

If the centre of mass is at the origin , R = 0 OR $\int r dm = 0$

- **Law of conservation of angular momentum**

If the external torque acting on a body is zero, the total angular momentum is conserved.

Mathematically, if $\tau_{external} = 0$, $\frac{dL}{dt} = 0$ OR L is a constant.

- **Linear momentum of a system of a system of particles**

Momentum , p = mv

Consider a system of n particles . Momentum of the particle of mass m_1 , $p_1 = m_1 v_1$

Momentum of the particle of mass m_2 , $p_2 = m_2 v_2$

Linear momentum of the system of particles , $P = m_1 v_1 + m_2 v_2 + m_3 v_3 + \ldots = p_1 + p_2 + p_3 \ldots \ldots$

or P = MV

ie, , total momentum of the system of particles is the vectors sum of momenta of all particles.

- **Newton's second law is applicable for a system of n particles**

Total momentum of an n particle system, P = MV

$\frac{dp}{dt} = M\frac{dv}{dt} = MA = F_{external}$

Thus Newton's second law is applicable for a system of n particles.

- **Law of conservation of momentum of n particle system**

For an n particle system, $F_{external} = \frac{dp}{dt}$

If $F_{external} = 0$, $\frac{dp}{dt} = 0$ OR P is a constant . Thus law of conservation of momentum is applicable to n particle system.

- **Equilibrium of a rigid body**

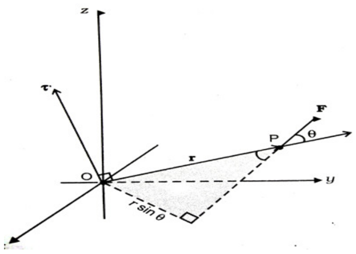
A body is in equilibrium if the net force and net torque acting on the body is zero.
If the force is zero , the total linear momentum is constant .
If the torque is zero , the total angular momentum is constant.
ie, For a body in equilibrium, a= 0 and $\propto$ = 0.

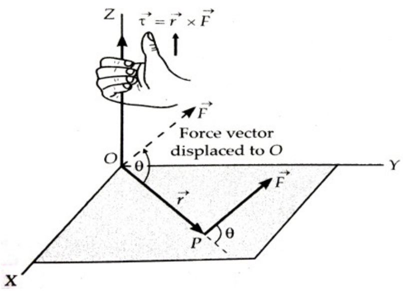
- Angular velocity (ω) at a particular instant is the same for all particles of a rotating body

  $\omega = \frac{d\theta}{dt}$ It is a vector quantity. Its direction is along the axis of rotation . It points out in the direction in which a right handed screw advances if its head rotates with the body.

  Linear velocity, v = $\vec{\omega} \times \vec{r}$

- Angular acceleration, $\propto = \frac{d\omega}{dt}$
- **Moment of force (torque)**





Torque about a point is the turning or rotating effect of force about that point.

Consider a force $\vec{F}$ acting at a point A. If $\vec{r}$ is the position vector of point A with respect to an origin O. Then torque about the point is $\vec{\tau} = \vec{r} \times \vec{F}$. It is a vector quantity.

Its unit is Nm (Newton metre).

Its direction is given by right handed screw rule.

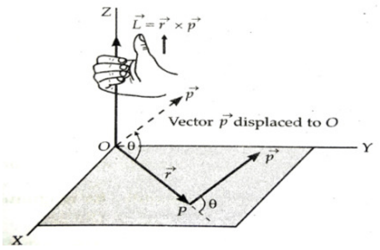
If the turning effect of force is clockwise, the torque is negative. If the turning effect of force is anti-clockwise, the torque is positive.

- **Angular momentum (moment of linear momentum)**

Angular momentum of a particle about a point is the moment of linear momentum about that point.

Angular momentum, $\vec{L} = \vec{r} \times \vec{p}$

It s unit is kg m^2/s. It is a vector quantity. Its direction is given by right handed screw rule.



- *Angular momentum,* $\vec{L} = \vec{r} \times \vec{P}$

  $\frac{d\vec{L}}{dt} = \frac{d\,(\vec{r} \times \vec{P})}{dt} = \frac{d\vec{r}}{dt} \times \vec{P} + \vec{r} \times \frac{d\vec{P}}{dt}$

  $\frac{d\vec{r}}{dt} \times \vec{P} = \frac{d\vec{r}}{dt} \times (m\vec{v}) = \vec{v} \times m\vec{v} = m\,\vec{v} \times \vec{v} = 0$

  $\therefore \frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{P}}{dt}$

  But $\frac{d\vec{P}}{dt} = \vec{F}$

  $\therefore \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$

- In planetary motion, the direction of gravitational force between the sun and the planet is along the radius. The angle between
  $\vec{F}$ and $\vec{r}$ is zero.

$\therefore \vec{\tau} = \vec{r} \times \vec{F} = 0$

- **Equation of motion for a rotatingbody**

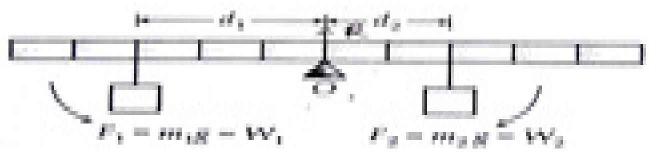
  $\omega = \omega_0 + \propto t$ ……..(1)

  $\theta = \omega_0 t + \frac{1}{2} \propto t^2$ ……….(2)

  $\omega^2 - \omega_0^2 = 2 \propto \theta$ ……(3)

- The load displaced using a lever is called resistance (load).
  The force applied on the lever is called effort.
  The point with respect to which a lever turns is called fulcrum.
  The perpendicular distance between effort and fulcrum is called effort arm.
  The perpendicular distance between load and fulcrum is called load arm.

- **For a see –saw**



Condition for translation equilibrium: R- $F_1 - F_2$=0

where F_1 and F_2 are the forces at the ends and R is the reaction of fulcrum.

Condition for rotational equilibrium:

$F_1 d_1 + -F_2 d_2 = 0$ (As the torques are in the opposite sense)

ie, total torque = 0 or $F_1 d_1 = F_2 d_2$

- **Mechanical advantage of a lever**

  It is the ratio of resistance (load) to effort.

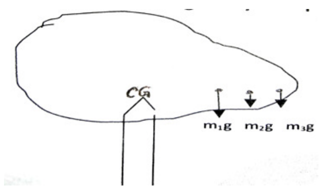
  or It is the ratio of effort arm to load to arm. Mechanical advantage (M.A) = $\frac{Load}{Effort}$ = $\frac{Effort\ arm}{Load\ arm}$

  or load $\times$ Load arm = Effort $\times$ $Effort\ arm$

  This is known as the principle of moments. (Principle of lever).

- *First order lever*: Fulcrum is in between load and effort.
  eg : Scissors , see-saw, common balance etc.
  Second order lever: Load (resistance) is in between effort and fulcrum.
  eg: nut cracker , lemon squeezer etc.
  Third order lever: Effort is in between resistance and fulcrum.
  eg: Ice tongs , fire tongs etc.

- **Centre of gravity**



Centre of gravity of a body is the point where the net gravitational torque on the body is zero .The body is in translational equilibrium and rotational equilibrium about centre of gravity.

Consider a body which is resting on its centre of gravity. Let the origin be at centre of gravity. The position vectors of particles of masses m_1, m_2, m_3 …… are $\vec{r_1}, \vec{r_2}, \vec{r_3}$ …….

Weight of the particles (force) are $m_1 g, m_2 g, m_3 g$…..

The torque are $\vec{r_1} \times m_1 g, \vec{r_2} \times m_2 g$, $\vec{r_3} \times m_3 g$

Total torque = $\sum \vec{r_i} \times m_1 g$

For a body resting on centre of gravity

τ = 0 OR $\sum \vec{r_i} \times m_1 g$ =0 ∴ g = 0

For a small body 'g' on all particles is the same

∴ g $\sum m_i r_i$ = 0 or $\sum m_i r_i$ = 0 OR $\int dmr$ = 0

ie, the centre of mass coincides with the origin. Here origin is the centre of gravity.

Centre of mass and centre of gravity coincides in gravity free region (g = 0) or when g is constant.

NB: For an extended body 'g' is not a constant.

- Moment of inertia is the property of a rotating body by virtue of which it resists the change in rotational motion. Unlike mass, moment of inertia is not fixed. It depends on the orientation and position of the axis of rotation.

- **Radius of gyration**

  Radius of gyration about an axis of rotation is the distance from the axis to a mass point whose mass is equal to the mass of the whole body and whose moment of inertia equal to the moment of inertia of the body about the axis.

Moment of inertia, $I = Mk^2$ where k is the radius of gyration.

For a ring, $I = MR^2$

Here $k^2 = R^2$

OR k = R

For a disc, $I = \frac{MR^2}{2}$

Here $k^2 = \frac{R^2}{2}$

OR $k = \frac{R}{\sqrt{2}}$

- Flywheel is a disc having large moment of inertia used in machines . As it has large moment of inertia, it can resist the sudden increase or decrease of the speed of the vehicle. It allows gradual change in speed and prevents jerky motion.

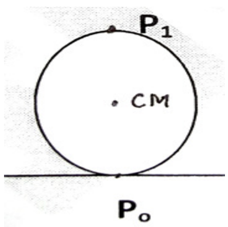
- Work done by torque, dw = $\tau d\theta$

  Power, P = $\frac{\tau d\theta}{dt} = \tau\omega$

- **Example for conservation of angular momentum**

  Consider a person sitting on a revolving chair. Angular momentum of the person, L = $I\omega$ where I is the moment of inertia and ω is the angular velocity. When the angular velocity increases, the person extends hands. Moment of inertia increases and ω decreases. Hence the angular momentum remains constant.

- **Rolling motion of a disc along a horizontal surface – condition for rolling without slipping**



Velocity of the point P_0 which is in contact with the ground = 0

Velocity of the centre of mass = R ω where R is the radius of the disc .

Velocity of the point P_1 = 2 R ω

- **Kinetic energy of a rolling body**

Kinetic energy = Kinetic energy of translation + Kinetic energy of rotation

Kinetic energy = $\frac{1}{2}MV_{cm}^2 + \frac{1}{2}\ I\ \omega_{cm}^2$

But $I = Mk^2$

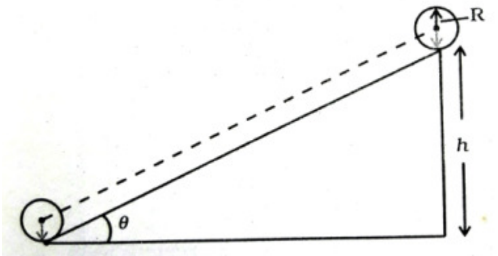
Kinetic energy = $\frac{1}{2}MV_{cm}^2 + \frac{1}{2}Mk^2\omega_{cm}^2$

$V_{cm} = \mathrm{R}\,\omega_{cm}$ OR $\omega_{cm} = \frac{V_{cm}}{R}$

Kinetic energy = $\frac{1}{2}MV_{cm}^2 + \frac{1}{2}Mk^2\left[\frac{V_{cm}}{R}\right]^2$

Kinetic energy = $\frac{1}{2}MV_{cm}^2\left[1 + \frac{k^2}{R^2}\right]$

- **A ring , solid cylinder and a sphere of equal radii are allowed to roll along an inclined plane of height h . Obtain the velocity with which they reach the bottom (all rolling without slipping).**



Potential energy at height h becomes the total kinetic energy during motion.

Potential energy at height h = Total kinetic energy

$Mgh = \frac{1}{2} MV_{cm}^2 \left[1 + \frac{k^2}{R^2}\right]$

$V_{cm}^2 = \frac{2gh}{\left[1+\frac{k^2}{R^2}\right]}$ OR $V_{cm} = \sqrt{\frac{2gh}{\left[1+\frac{k^2}{R^2}\right]}}$

<u>**Ring :**</u> Moment of inertia , $I = MR^2$ Also $I = Mk^2 \therefore k^2 = R^2$

$V_{cm} = \sqrt{\frac{2gh}{\left[1+\frac{R^2}{R^2}\right]}} = \sqrt{gh}$

<u>**Solid cylinder**</u> **:** Moment of inertia , $I = \frac{M\,R^2}{2} \therefore k^2 = \frac{R^2}{2}$

$V_{cm} = \sqrt{\frac{2gh}{\left[1+\frac{R^2}{2R^2}\right]}} = \sqrt{\frac{4gh}{3}}$

<u>**Sphere**</u> **:** Moment of inertia, $I = \frac{2M\,R^2}{5} \therefore k^2 = \frac{2R^2}{5}$

$V_{cm} = \sqrt{\frac{2gh}{\left[1+\frac{2R^2}{5R^2}\right]}} = \sqrt{\frac{10\,gh}{7}}$. Sphere attains maximum velocity.