8

GRAVITATION

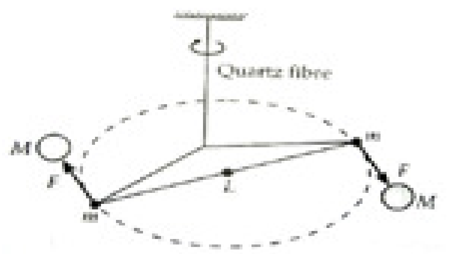
- **Newton's universal law of gravitation and its mathematical form**

"Every body in the universe attracts every other body with a force. This force is directly proportional to the product of their masses and inversely proportional to the square of distance between them."
Consider two bodies of masses m_1 and m_2. Their centres are separated by a distance of r.
The force of attraction between them,
$F \propto m_1 m_2$

$$F \propto \frac{1}{r^2}$$

OR $F \propto \frac{m_1 m_2}{r^2}$ or $F = G\frac{m_1 m_2}{r^2}$ where G is the constant of gravitation. G = 6.67$\times 10^{-11} Nm^2/kg^2$.
The value of G was determined by Cavendish.

- **Cavendish's experiment for the determination of G**



The experimental set up consists of two small lead balls, each of mass m, are connected to the ends of a light rod of length L. The rod is supported by a vertical quartz fibre. Two large lead spheres, each of mass M, are placed near the ends of small balls on opposites sides. The small spheres are attracted towards the larger ones by a force, $F = \frac{\mathrm{GMm}}{r^2}$

The force on the two small spheres form a couple .

This torque deflects the rod and twists the suspension fibre.

The angle of deflection θ is measured using a lamp and scale arrangement .

Deflecting torque = $F \times L = \frac{\mathrm{GMm}}{r^2} L$

Restoring torque = $k\theta$

where k is the restoring torque per unit twist or torsion constant of the suspension fibre.

In rotational equilibrium ,

$\frac{GMm}{r^2} L = k\theta$ or $G = \frac{k\theta r^2}{MmL}$

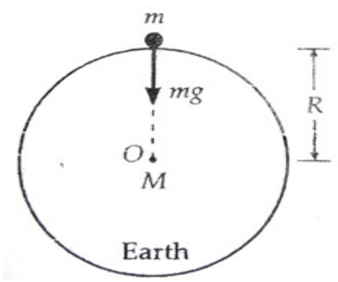
Thus, G is determined.

- The dimensional formula for 'G' is $M^{-1}L^3T^{-2}$.

- **Acceleration due to gravity**

The acceleration produced on a freely falling body due to the gravitational pull of the earth is called acceleration due to gravity(g).

- **Expression for acceleration due to gravity**



Consider a body of mass m near the surface of the earth.

The gravitational force exerted by the earth on it, F = mg --------------(1)

According to Newton's law of gravitation $F = G\frac{Mm}{r^2}$ --------------(2) where M is the mass of the earth.

mg = $G\frac{Mm}{r^2}$ or g = $\frac{GM}{r^2}$

If the body is very close to the earth's surface, r = R

∴ g = $\frac{GM}{R^2}$.

This is the expression for acceleration due to gravity.

Clearly, acceleration due to gravity is independent of the mass of the body.

$g \propto \frac{1}{R^2}$

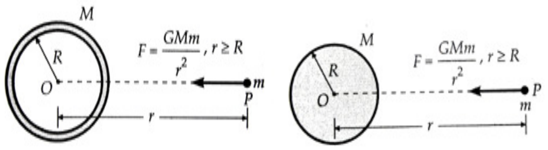
Polar radius is smaller than equatorial radius.

∴g is more at poles than at equator.

- **Newton's shell theorem**.

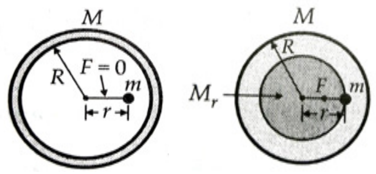
It gives gravitational force on a point mass due to a spherical shell or a solid sphere.

❖ If a point mass lies outside a uniform spherical shell or sphere with a spherically symmetric internal mass distribution , the shell or sphere attracts the point mass as if the entire mass of the shell or sphere were concentrated at its centre.

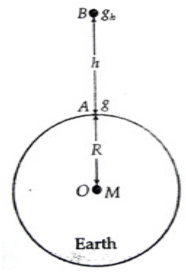


❖ If a point mass lies inside a uniform spherical shell, the gravitational force on the point mass is zero.

But if a point mass is inside a homogeneous solid sphere, the force on the point mass acts towards the centre of the sphere. This force is exerted by the spherical mass interior to the point mass.



- **Variation of acceleration due to gravity with altitude**



On the surface of the earth, $g = \frac{GM}{R^2}$ ....(1) where M is the mass of the earth and R is the radius.

At a height h above the surface of earth, $g_h = \frac{GM}{(R+h)^2}$ ....(2)

(1) ÷ (2)

$\frac{g}{g_h} = \frac{GM}{R^2} \div \frac{GM}{(R+h)^2}$

$$\frac{g}{g_h} = \frac{GM}{R^2} \times \frac{(R+h)^2}{GM}$$

$\frac{g}{g_h} = \frac{(R+h)^2}{R^2}$

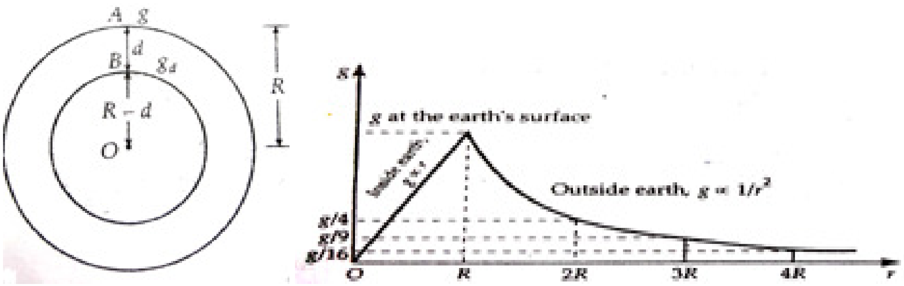
$\frac{g}{g_h} = \left[\frac{R+h}{R}\right]^2$ OR $\frac{g}{g_h} = \left(1+\frac{h}{R}\right)^2$

$g = g_h\left(1+\frac{h}{R}\right)^2$ OR $g_h = \frac{g}{\left(1+\frac{h}{R}\right)^2}$

$g_h = g\left(1+\frac{h}{R}\right)^{-2}$

OR $g_h = g\left(1-\frac{2h}{R}\right)$ (by binomial approximation) ie,Acceleration due to gravity decreases with increase of altitude.

- **Variation of acceleration due to gravity with depth**



The acceleration due to gravity on the surface of earth, g = $\frac{GM}{R^2}$ …………..(1)

where M is the mass of the earth and R is the radius.

Consider a body of mass 'm' at a depth d. The gravitational force on 'm' due to the shell of thickness d will be zero.

The gravitational force on 'm' is only due to the part of the earth of radius (R-d).

The acceleration due to gravity on 'm' is $g_d = \frac{GM'}{(R-d)^2}$ ……………..(2) where M' is the mass of the sphere of radius (R-d).

Density = $\frac{mass}{volume}$ or Mass = Density × Volume

If ρ is the density of the earth, M = $\rho \frac{4}{3} \pi R^3$

M' = $\rho \frac{4}{3} \pi (R-d)^3$ (ρ is the same in both cases as the material of the earth is the same).

Putting these in (1) and (2)

g = $\frac{G}{R^2} \times \frac{4}{3} \pi R^3 \rho$

g = $\frac{4}{3} G\pi R\rho$ …………….(3)

$$g_d = \frac{G}{(R-d)^2} \times \frac{4}{3} \pi (R-d)^3 \rho$$

$g_d = \frac{4}{3} G\pi (R-d)\rho$ …………….(4)

(4) ÷ (3)

$\frac{g_d}{g} = \frac{\frac{4}{3} G\pi(R-d)\rho}{\frac{4}{3} G\pi R\rho}$ OR $\frac{g_d}{g} = \frac{(R-d)}{R}$

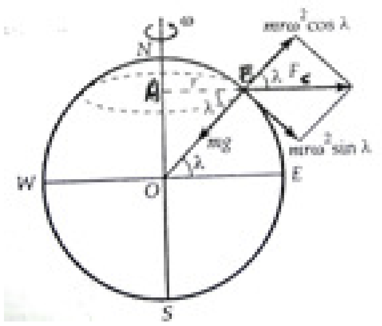
$$\frac{g_d}{g} = 1 - \frac{d}{R} g_d = g\left(1 - \frac{d}{R}\right)$$

The acceleration due to gravity decreases with the increase of depth.
At the centre of the earth d = R

$\therefore g_d = g\left(1 - \frac{d}{R}\right) = 0$

$\therefore$ The body will be weightless at the centre of the earth.

- **Variation of acceleration due to gravity with latitude**



Consider a body of mass 'm' at a latitude λ. When the earth rotates about its axis, the body rotates in a circle of radius 'r'. It experiences a centrifugal force, $F_c = \frac{mv^2}{r} = \frac{m}{r}(r\omega)^2 = m\,r\,\omega^2$ (outwards).

The body will be attracted towards the centre of the earth with a force mg.

$F_c \cos\lambda$acts in the opposite direction.

The net force on the body = mg - $F_c \cos\lambda$

mg_l= mg – $m\,r\omega^2 \cos\lambda\, g_l$ = g - $r\omega^2 \cos\lambda$From $\Delta\, OAB, \cos\lambda$ = r/R or r = R $\cos\lambda\, g_l$ = g – R $\cos\lambda\, \omega^2 \cos\lambda\, g_l$ = g – R $\omega^2 \cos^2\lambda$

At poles λ = 90^0 At equator λ = 0^0

- Gravitational force is a **conservative force** because the work done by the force in moving a body between two points depends on **the initial and final positions of the body only**. It does not depend on the path covered by the body.

- Gravitational force is a **central force** as it acts along the line joining the centres of two masses.

- **Mass of the earth**

We know that, acceleration due to gravity on the surface of the earth, g = $\frac{GM}{R^2}$ or GM = gR^2

or M = $\frac{gR^2}{G}$

- **Density of the earth**

Density = $\frac{mass}{volume}$(we assume that the earth is a perfect sphere.)

Density = $\frac{M}{\frac{4}{3}\pi R^3} = \frac{gR^2}{G} \div \frac{4}{3} \pi R^3$

$$= \frac{gR^2}{G} \times \frac{3}{4\pi R^3}$$

$\rho = \frac{3g}{4\pi RG}$

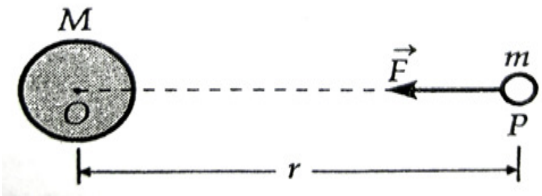
- **Gravitational field of a body**

It is the space surrounding a body where its gravitational force is felt.

- **Intensity of gravitational field at a point**

Intensity of gravitational field at a point is the force experienced by a unit mass kept at that point. It is a vector quantity. Its direction is towards the mass producing gravitational field. Its unit is N/m.

- **Expression for the intensity of gravitational field at a point**



Consider a body of mass M. P is a point at a distance ‘r’ from it. To find the gravitational field intensity at P, we keep a unit mass (m) at P. The force on m is the intensity.

$F = \frac{GMm}{r^2}$ Here m = 1 kg

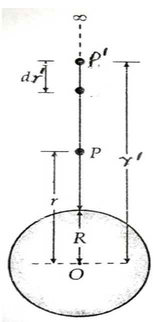
∴ Intensity of gravitational field at P $= \frac{GM}{r^2}$

If ‘r’ is more, intensity of gravitational field is less.

- **Gravitational potential**

Gravitational potential at any point is defined as the work done in bringing a unit mass from infinity to that point without accelerating it.

- **Gravitational potential at a point**



Consider a body of mass M. P is a point a distance r from it. The potential at P is the work done in bringing a unit mass from infinity to that point. On the way consider a point P’ at a distance r’ from M.

Gravitational force on m at P’ is $F = \frac{GMm}{r'^2}$

Work done to move m through small path of length dr’ at P’,

dW = magnitude of applied force x displacement dW = $\frac{GMm}{r'^2} \times$ dr’

The work done in bringing from infinite separation to r separation is

W = $\int_{\propto}^{r} \frac{GMm}{r'^2}$ dr’

= GMm$\int_{\propto}^{r} \frac{1}{r'^2}$ dr’

= GMm$\int_{\propto}^{r} (r')^{-2}$dr’

= GMm$\left[\frac{(r')^{-2+1}}{-2+1}\right]_{\propto}^{r}$

= GMm$\left[\frac{-1}{r'}\right]_{\propto}^{r}$

$= -\, GMm\left[\frac{1}{r} - \frac{1}{\propto}\right]$

But $\frac{1}{\propto}$ = 0

∴ W = $\frac{-GMm}{r}$

Here m= 1 ∴ W = $\frac{-GM}{r}$

This work is the potential at P. ∴ V = $\frac{-GM}{r}$As ‘r’ increases, V increases. Its maximum value is at infinity.

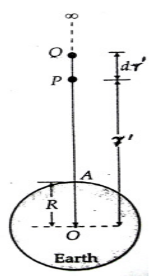
In the gravitational field, V is negative.

- **Escape velocity from the earth**

It is a minimum velocity required to escape from the gravitational pull of the earth.

- The escape velocity from the earth is 11.2 km/s.
- The escape velocity from the moon is 2.38 km/s.

- **Escape velocity from the earth**



Consider a body of mass ‘m’ at a distance r from the centre of the earth whose mass is M and radius is R.

The gravitational force exerted by the earth on the body = $\frac{GMm}{r^2}$

To move the body against this force along a small path length dr is dw = applied force × displacement

dw = $\frac{GMm}{r^2}$ dr

Work done in moving the body from the surface of earth to infinity is

$W = \int_R^\alpha \frac{GMm}{r^2}$ dr

$= GMm \int_R^\alpha \frac{1}{r^2}$ dr

$= GMm \int r^{-2}$ dr

$= GMm \times \left[\frac{r^{-2+1}}{-2+1}\right] \begin{matrix}\alpha \\ R\end{matrix}$

$= \ GMm \left[\frac{r^{-1}}{-1}\right] \begin{matrix}\alpha \\ R\end{matrix}$

$= -GMm[r^{-1}]_R^\alpha$

$= -\ GMm \left[\frac{1}{r}\right] \begin{matrix}\alpha \\ R\end{matrix}$

$= -GMm \left[\frac{1}{\alpha} - \frac{1}{R}\right]$

But $\frac{1}{\alpha} = 0$

$\therefore$ W = $-$ GMm $\times \frac{-1}{R}$

$W = \frac{GMm}{R}$

This work is converted to the kinetic energy of the body

KE = $\frac{1}{2} m{v_e}^2$ where v_e is the escape velocity

$\therefore \frac{GMm}{R} = \frac{1}{2} m{v_e}^2$

$\frac{GM}{R} = \frac{{v_e}^2}{2}$

${v_e}^2 = \frac{2GM}{R}$

$v_e = \sqrt{\frac{2GM}{R}}$

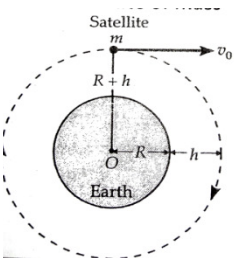
But GM = gR^2

$\therefore v_e = \sqrt{\frac{2gR^2}{R}} = \sqrt{2gR}$

This is the expression for the escape velocity .

- The path of a satellite around the earth is called its orbit.

- **Orbital velocity of satellite**



The minimum velocity required for a satellite to move in its orbit is called orbital velocity. Consider a satellite of mass ‘m’ revolving round the earth in an orbit of radius ‘r’.

The gravitational force exerted by the earth on it provides centripetal force for the satellite.

Gravitational force = centripetal force $\frac{GMm}{r^2} = \frac{mv_o^2}{r}$ where v_o is the orbital velocity.

$\frac{GM}{r} = v_o^2$ Or $v_0 = \sqrt{\frac{GM}{r}}$

But GM = gR^2

$\therefore v_0 = \sqrt{\frac{gR^2}{r}} = \sqrt{\frac{gR^2}{R+h}}$

h $\ll R$ ∴ h is neglected.

$\therefore v_0 = \sqrt{gR}$

Clearly, escape velocity = $\sqrt{2}v_0$

- Velocity of a satellite in its minimum orbit is called first cosmic velocity. It is about 7.8km/s.

Velocity of a body to escape from gravitational pull of earth is called second cosmic velocity.
It is about 11.2 km/s.

Velocity of a body to escape from solar system is called third cosmic velocity. It is about 16.7 km/s.

Velocity of a body to escape from milky way is called fourth cosmic velocity. It is about 130 km/s.

- **Gravitational potential energy**

Consider a point P at a distance 'r' from a body of mass M. The gravitational potential at P is $\frac{-GM}{r}$When a body of mass m is kept at P, it will be attracted by M.

Potential = $\frac{Work}{mass}$

Work = potential $\times$ $mass$

Gravitational potential energy = Gravitational potential $\times$ $mass$ $= \frac{-GM}{r} \times m = \frac{-GMm}{r}$

- **Prove that the work done in lifting a body to a height h above the earth is mgh.**

Consider a body of mass m on the surface of earth whose mass is M.

Gravitational P.E on the surface = $\frac{-\text{GMm}}{\text{r}}$ Here r = R Gravitational P.E on the surface = $\frac{-GMm}{R}$

When the body is lifted to a height h, the new gravitational P.E = $\frac{-GMm}{R+h}$

Difference in gravitational P.E = Work done in lifting the body $= \frac{-GMm}{R+h} - \frac{-GMm}{R}$ = - GMm $\left[\frac{1}{R+h} - \frac{1}{R}\right]$ = - GMm $\left[\frac{R-(R+h)}{(R+h)R}\right]$ = - GMm $\left[\frac{R-R-h}{R^2+Rh}\right]$ = - GMm$\times \frac{-h}{R^2+Rh}$ $= \frac{GMmh}{R^2[1+\frac{h}{R}]}$ $\frac{h}{R}$ is very small ∴ it is neglected. ∴ Work done $= \frac{\text{GMmh}}{\text{R}^2}$ But $\frac{GM}{R^2}$ = g

∴ Work done = gmh = mgh

- **Time period of a satellite**

The time taken by a satellite to move once in its orbit is called time period.

Speed = $\frac{\text{distance}}{\text{time}}$

Time = $\frac{\text{distance}}{\text{speed}}$

Time period $= \frac{2\pi r}{v_o} = \frac{2\pi r}{\sqrt{\frac{GM}{r}}} = 2\pi r\sqrt{\frac{r}{GM}}$

$T = 2\pi\sqrt{\frac{r^3}{GM}}$ where r = R+ h

- **Total energy of a satellite**.

  Consider a satellite of mass m moving around the earth with velocity v_0 in an orbit of radius r. The satellite possesses potential energy due to the gravitational pull of the earth.

  $$U = -\frac{GMm}{r}$$

  The kinetic energy of a satellite,

  $$K = \frac{1}{2}mv_0^2 = \frac{1}{2}m\left(\frac{GM}{r}\right)$$

  Total energy of the satellite is

  $E = U + K = -\frac{GMm}{r} + \frac{1}{2}\frac{GMm}{r}$ or $E = \frac{-GMm}{2r} = -\frac{GMm}{2(R+h)}$

  The total energy is negative . It shows that the satellite is bound to the earth by strong gravitational attraction .

- **Binding energy of a satellite** .

  The energy required by a satellite to leave its orbit around the earth and escape to infinity is called its binding energy.

  The total energy of a satellite = $-\frac{GMm}{2r}$

  In order to escape to infinity , it must be given an extra energy of $+\frac{GMm}{2r}$ such that its total energy becomes equal to zero.

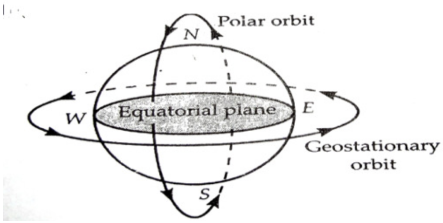
  Binding energy of a satellite $= \frac{GMm}{2r} = \frac{GMm}{2(R+h)}$

- **Geostationary satellite Or synchronous satellites**

  Geostationary satellites take 24 hours for one revolution around the earth. As the time period of rotation of earth and the time period of revolution of satellite are equal they appear to be stationary on observing from the earth. As the time periods are equal they are also called synchronous satellites. They orbit around the earth over the equator at a distance of about 36000 km from the surface of the earth. The orbit of geostationary satellite is called **parking orbit**. They are mainly used for telecommunication.

- **Polar satellites**

  Polar satellites orbit around the earth in orbits around poles. They are launched at a distance of 500 – 800 km from the surface of the earth. They are used for weather forecasting, environmental studies, locating mineral sources and spy work. Their time period of revolution is small (about 100 minute). As the earth rotates under it, the observation of vast area on earth is possible.



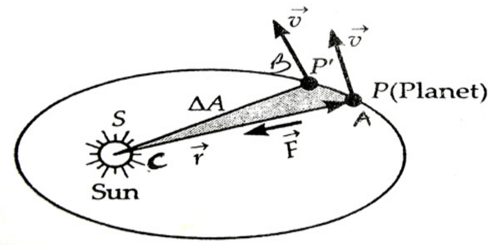
- Geocentric model of solar system was suggested by Ptolomy.

- Copernicus gave the heliocentric model of solar system.

- **Kepler's laws of planetary motion**

1. Law of orbits: The orbit of a planet is an ellipse with the sun at one of its foci.
2. Law of areas: The line joining the planet to the sun sweeps equal areas in equal intervals of time. OR Areal . velocity of a planet revolving round the sun is a constant
3. Law of periods OR harmonic Law: The square of time period of planet is proportional to the cube of semi major axis of the ellipse followed by the planet.

- **Proof of Kepler's second Law**



Consider a planet revolving round the sun. Initially it is at A with position vector $\vec{r}$.In timeΔ t ,it reaches a new position B with position vector$\vec{r}+\overrightarrow{\Delta r}$

From triangle law of vectors AB = Δr.

Area of Δ ABC, $\Delta A = \frac{1}{2}$ bh = $\frac{1}{2}$ r $\Delta r \frac{\Delta A}{\Delta t}$ = $\frac{1}{2} r \frac{\Delta r}{\Delta t}$

When Lt $\to$ $0 \frac{dA}{dt} = \frac{1}{2} r \frac{dr}{dt}$

$$\frac{dA}{dt} = \frac{1}{2} rv$$

But magnitude of momentum, L= mvr∴ rv = $\frac{L}{m}$

∴ $\frac{dA}{dt}$ = $\frac{1}{2}\frac{L}{m}$ L is constant in planetary motion. $\frac{1}{2}$and m are constants. ∴ $\frac{dA}{dt}$ = constant

i.e. Areal velocity = constant which is second Law.

- **Proof of Kepler's third law**

To prove this we assume the orbit to be circular.

Time period of planet,T = $\frac{2\pi r}{v_0}$

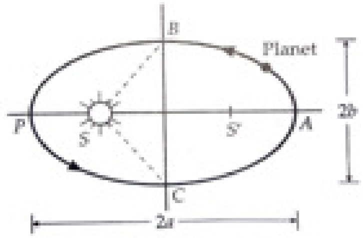
= $\frac{2\pi r}{\sqrt{\frac{GM}{r}}}$

$$= 2\pi r\sqrt{\frac{r}{GM}}$$

T = $2\pi\sqrt{\frac{r^3}{GM}}$ Where r = R+ h

Squaring $T^2 = 4\pi^2 \frac{r^3}{GM}$ OR $T^2 \propto r^3$ which is third law.

- The points P and A on the orbit are called the **perihelion** and the **aphelion** and they represent the closest and farthest distances from the sun respectively.



- **Reason for weightlessness in artificial satellite**

  The weight of bodies in satellite is completely used for providing centripetal force. Hence the bodies experience weightlessness.